Simultaneous equations

Matrices can be used to solve simultaneous linear equations, by first writing them in matrix form and then pre-multiplying by the inverse.

Example (Method 1)

Solve the simultaneous equations (using matrices):

$$2x-3y=14$$
$$3x+4y=4$$

Solution

Note firstly that these simultaneous equations can be written as the following matrix equation. If you need convincing, multiply out the matrices.

$$\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 14 \\ 4 \end{pmatrix}$$

Pre-multiplying by the inverse matrix on both sides, we get:

$$\frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \end{pmatrix}$$

This gives the identity matrix on the left hand side, so we simplify to get:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{17}\begin{pmatrix} 68 \\ -34 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

So the solution is $x=4$ and $y=-2$.

Question 1.1

Solve the simultaneous equations (using matrices):

$$2p - 4q = -14$$

$$2q - 3p = 13$$

Matrices can also be used in a different way to solve simultaneous equations. This method more closely resembles the method used to solve them algebraically that we covered earlier in the course. We will look at the last example again, this time looking at an elimination method.

Example (Method 2)

Solve the simultaneous equations (using matrices):

$$2x-3y=14$$

$$3x+4y=4$$

Solution

We can write this in the abbreviated form:

$$\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 14 \\ 4 \end{pmatrix}$$

This can in turn be written as:

$$\left(\begin{array}{cc|c} 2 & -3 & 14 \\ 3 & 4 & 4 \end{array}\right)$$

We would like to get zero in the bottom left hand corner. We are allowed to replace any row with a linear combination of the rows. We will do this in two stages. Replacing the first row by second row minus first row ($(2)-(1)$):

$$\left(\begin{array}{cc|c} 1 & 7 & -10 \\ 3 & 4 & 4 \end{array}\right)$$

Replacing the second row by $(2)-3\times(1)$:

$$\left(\begin{array}{cc|c} 1 & 7 & -10 \\ 0 & -17 & 34 \end{array}\right)\Rightarrow\left(\begin{array}{cc|c} 1 & 7 & -10 \\ 0 & 1 & -2 \end{array}\right)$$

This tells us that $y=-2$, and so $x=-10-7y=4$.

Example

Solve the simultaneous equations (using the elimination method):

$$2x+y-2z=11$$

$$3x-2y+4z=-8$$

$$x+4y-3z=15$$

Solution

These can be written as:

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 11 \\ 3 & -2 & 4 & -8 \\ 1 & 4 & -3 & 15 \end{array}\right)$$

This time we would like to get zeros in the bottom left hand corner. Replacing the first row by $(1)-(3)$ and the second row by $(2)-3\times(3)$ we get:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -4 \\ 0 & -14 & 13 & -53 \\ 1 & 4 & -3 & 15 \end{array}\right)$$

Replacing the third row by $2\times((3)-(1))+(2)$, we get:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -4 \\ 0 & -14 & 13 & -53 \\ 0 & 0 & 5 & -15 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 1 & -4 \\ 0 & -14 & 13 & -53 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Replacing the second row by $(2)-13\times(3)$, we get:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -4 \\ 0 & -14 & 0 & -14 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

so $z=-3,\ y=1,\ x=2$.

When solving simultaneous equations, you do not always have as many different equations as it first appears. Any equation that can be written as a linear combination of the other equations (*ie* you can obtain this equation from the others by adding, subtracting or multiplying) is said to be linearly dependent on the others. When solving simultaneous equations you should ensure that the equations you are working with form a linearly independent set. You only need to solve linearly independent equations. For example, suppose you were given the simultaneous equations:

$$3x-4y=-5$$

$$2x+3y=8$$

$$x-7y=-13$$

Here you would only need to consider, say, the first two equations since the third is a linear combination of them (equation 3 is equation 1 minus equation 2). You could equally ignore the first or second equation. Careful choice of which equations to use can make things considerably easier for you, so do watch out.

Question 1.2

Solve the simultaneous equations:

$$3x+2y=13$$

$$7x-6y=9$$

using the method of the last example.

Question 1.3

Solve the simultaneous equations:

$$x+2y-z=-4$$

$$2x-3y+2z=1$$

$$3x+y-3z=-17$$

using the method of the last example.

Solutions

Solution 1.1

Noticing that the letters are in different orders, these equations can be written in the form:

$$\begin{pmatrix} 2 & -4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -14 \\ 13 \end{pmatrix}$$

Pre-multiplying both sides by the inverse of $\begin{pmatrix} 2 & -4 \\ -3 & 2 \end{pmatrix}$, we get:

$$\begin{pmatrix} p \\ q \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -14 \\ 13 \end{pmatrix}$$

$$= -\frac{1}{8}\begin{pmatrix} 24 \\ -16 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

So $p = -3$ and $q = 2$.

Solution 1.2

These equations can be written as:

$$\left(\begin{array}{cc|c} 3 & 2 & 13 \\ 7 & -6 & 9 \end{array}\right)$$

Replacing (1) by $(2) - 2 \times (1)$:

$$\left(\begin{array}{cc|c} 1 & -10 & -17 \\ 7 & -6 & 9 \end{array}\right)$$

Replacing (2) by $(2) - 7 \times (1)$:

$$\left(\begin{array}{cc|c} 1 & -10 & -17 \\ 0 & 64 & 128 \end{array}\right)$$

So $x = 3$ and $y = 2$.

Solution 1.3

These equations can be written as:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 2 & -3 & 2 & 1 \\ 3 & 1 & -3 & -17 \end{array}\right)$$

Replacing (2) by $(2)-2\times(1)$, and (3) by $(3)-(2)-(1)$, we get:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & -7 & 4 & 9 \\ 0 & 2 & -4 & -14 \end{array}\right)$$

Replacing (2) by $(2)+(3)$, we get:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & -5 & 0 & -5 \\ 0 & 2 & -4 & -14 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -4 & -14 \end{array}\right)$$

Replacing (3) by $(3)-2\times(2)$, we get:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

This gives $z=4,\ y=1,\ x=-2$.

